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On vector analogs of the modified Volterra lattice

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Abstract
The zero curvature representations, Bäcklund transformations, nonlinear superposition principle and the simplest explicit solutions of soliton and breather type are presented for two vector generalizations of modified Volterra lattice. The relations with some other integrable equations are established.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Vector equations are an important and rather well-studied class of integrable systems. In this area, we mention only a few works [1–3] containing the examples and classification results for the vectorial systems of derivative nonlinear Schrödinger type which are in some relation to the theme of our paper. There are also several interesting results for the vector differential–difference equations, or lattices, see e.g. [4, 5], but this field seems less investigated. The aim of our work is the study of the vector lattices

$$V_{n,x} = 2\langle V_n, V_{n+1} - V_{n-1}\rangle V_n - \langle V_n, V_n\rangle(V_{n+1} - V_{n-1}), \qquad (1)$$

$$V_{n,x} = \langle V_n, V_n\rangle(V_{n+1} - V_{n-1}), \qquad (2)$$

which define two integrable generalizations of the very well-known modified Volterra lattice. Equation (1) was introduced in [6] among the other examples of the multi-component lattices related to Jordan algebraic structures. Equation (2) was introduced in [7] under the name of generalized equation of nonlinear filters for the case of two-dimensional vectors and in [8, 9] for the vectors of arbitrary dimension.

The main tool in the study of a nonlinear integrable equation is its representation as the compatibility condition for auxiliary linear systems. In the differential–difference setting this method was developed in the classical papers [10, 11]. In our paper we restrict ourselves to the version of dressing method based on Darboux–Bäcklund transformations and their nonlinear superposition principle. Bäcklund transformations for scalar Volterra and modified Volterra lattices were studied by many authors, see e.g. [12–14]. The summary of the main results for the scalar case is given in section 2. The main body of the paper, sections 3, 4, contains generalizations of this method for both vector lattices (1) and (2), as well as the simplest explicit solutions of soliton and breather type. The comparison with the results obtained in [15] for the lattice (2) is given.

A characteristic feature of integrability is the consistency of the equation with an infinite hierarchy of other equations. In particular, Bäcklund transformations define the discrete part of this hierarchy and lead to the discrete equations on the square grid. Usually, one starts this way from the continuous equations of KdV type, however an understanding appeared recently that the lattice equations of Volterra type lead to the same result as well [16–18]. This relation has not yet been observed in the vector case, although the discrete equation related to the lattice (1) has been introduced in the paper [19], see also [20–22].

The continuous part of the picture (section 5) is more traditional. It was observed in the works of Levi [23] and Shabat, Yamilov [24] that integrable Volterra-type lattices define a special kind of Bäcklund transformation for equations of nonlinear Schrödinger type. This remains valid for the vector analogs as well. The connection with a two-dimensional lattice relative to the Volterra lattice introduced by Mikhailov [25] is of interest, too. Finally, it should be noted that the approach based on the continuous symmetries is the most effective one in the classification problem of integrable equations, both continuous and discrete [26]. The complete classification of scalar Volterra-type lattices was obtained by Yamilov [27] by the use of the symmetry approach, see also [28, 29]. Some progress in the classification of vector equations and lattices has been achieved recently [3, 30–32]. We discuss some open problems in this field in section 6.

2. Scalar case

2.1. Zero curvature representations

The following notations for the auxiliary linear equations are used throughout the paper:

$$\Psi_{n+1} = L_n\Psi_n, \qquad \Psi_{n,x} = A_n\Psi_n, \qquad \tilde\Psi_n = M_n\Psi_n. \tag{3}$$

The modified Volterra lattice

$$v_{n,x} = \left(v_n^2 + a\right)(v_{n+1} - v_{n-1}) \tag{4}$$

is equivalent to the compatibility condition $L_{n,x} = A_{n+1}L_n - L_nA_n$ with the matrices

$$L_n = \begin{pmatrix} \frac{a}{\lambda} & v_n \\ -v_n & \lambda \end{pmatrix}, \qquad A_n = \begin{pmatrix} \frac{a^2}{\lambda^2} + v_{n-1}v_n & \frac{a}{\lambda}v_n + \lambda v_{n-1} \\ -\frac{a}{\lambda}v_{n-1} - \lambda v_n & \lambda^2 + v_{n-1}v_n \end{pmatrix}. \tag{5}$$

The Darboux–Bäcklund transformation is defined by the matrix

$$M_n = \frac{1}{a + \mu^2 f_n^2} \begin{pmatrix} \mu(a^2 - \mu^2\lambda^2) - a\mu(\lambda^2 - \mu^2)f_n^2 & -(a^2 - \mu^4)\lambda f_n \\ (a^2 - \mu^4)\lambda f_n & a\mu(\lambda^2 - \mu^2) - \mu(a^2 - \mu^2\lambda^2)f_n^2 \end{pmatrix}. \tag{6}$$

Moreover, the compatibility condition $\tilde L_n M_n = M_{n+1}L_n$ is equivalent to the pair of discrete Riccati equations for the variable f_n:

$$v_n = \frac{\mu f_{n+1} - af_n/\mu}{1 + f_nf_{n+1}}, \qquad \tilde v_n = \frac{\mu f_n - af_{n+1}/\mu}{1 + f_nf_{n+1}} \tag{7}$$

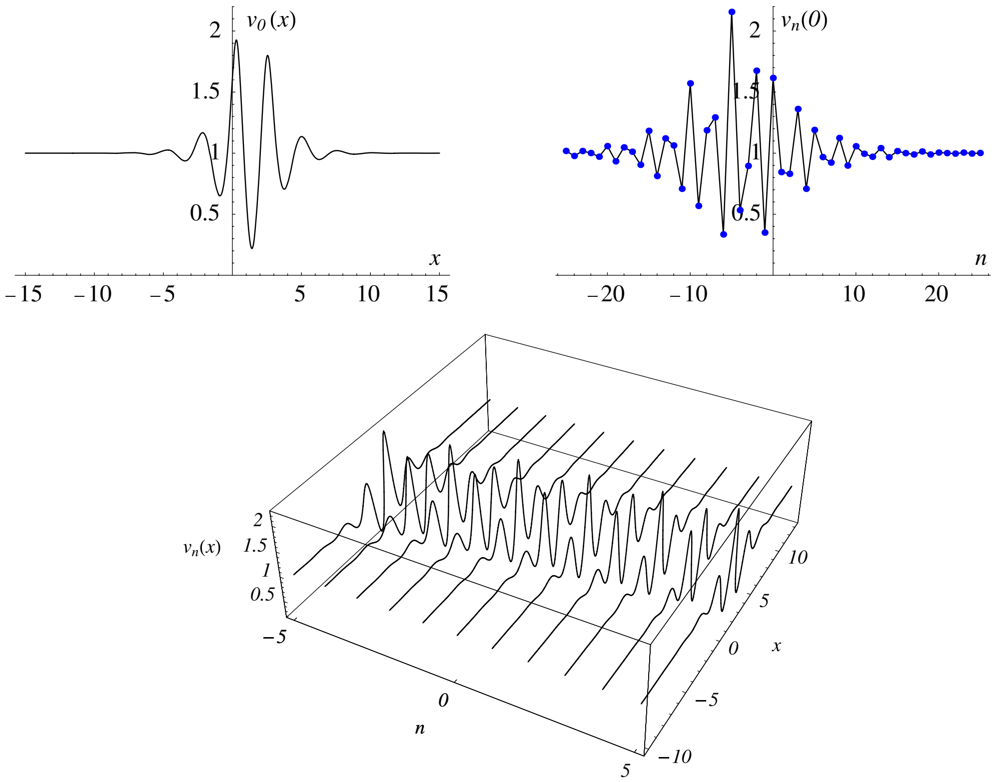

Figure 1. Breather of the lattice (4); $a = 1$, $\gamma_1^{(1)} = \bar\gamma_1^{(2)} = 0.5 + 1.5\mathrm{i}$, $k^{(1)} = k^{(2)} = 1$.

and the condition $M_{n,x} = \tilde A_n M_n - M_n A_n$ completes this system with the continuous Riccati equation

$$f_{n,x} = \left(\frac{a}{\mu} v_{n-1} + \mu v_n\right) f_n^2 + \left(\frac{a^2}{\mu^2} - \mu^2\right) f_n + \mu v_{n-1} + \frac{a}{\mu} v_n. \tag{8}$$

Note also that the variable f_n satisfies, in virtue of equations (7) and (8), the lattice

$$f_{n,x} = \frac{\left(\mu^2 + a f_n^2\right)\left(a + \mu^2 f_n^2\right)(f_{n+1} - f_{n-1})}{\mu^2(1 + f_{n+1} f_n)(1 + f_n f_{n-1})}. \tag{9}$$

Starting from a known solution v_n of the lattice (4) the common solution of the first equations (7) and (8) is constructed by the formula $f_n = \phi_n/\varphi_n$ where $\Psi = (\phi, \varphi)$ is a particular solution of two first equations (3) at $\lambda = \mu$. Then the second equation (7) defines the new solution $\tilde v_n$.

For example, in order to construct solutions of soliton type one takes $v_n = 1$ as the seed solution (obviously, the choice of another constant is equivalent to scaling of parameter a; some generalization can be achieved via dressing of the solution $v_{2n} = \alpha$, $v_{2n+1} = \beta$). The eigenvalues of the matrix $L_n|_{v_n=1,\lambda=\mu}$ are defined from the equations

$$\gamma_1 + \gamma_2 = \mu + a/\mu, \qquad \gamma_1\gamma_2 = 1 + a$$

and the corresponding solution of the linear equations is

$$\varphi_n = \gamma_1^n \,\mathrm{e}^{(\gamma_1^2+2)x} + k\gamma_2^n \,\mathrm{e}^{(\gamma_2^2+2)x}, \qquad \phi_n = \mu\varphi_n - \varphi_{n+1}$$

(we do not consider the case of multiple roots $\gamma_1 = \gamma_2$ which leads to rational in n, x solutions). The ratio $f_n = \phi_n/\varphi_n$ defines the solution of the lattice (9) of kink type (provided $\gamma_1/\gamma_2 > 0, k > 0$) and the substitution into the second equation (7) gives the soliton of the lattice (4). The construction of N-soliton solution uses the set of particular solutions $\big(\phi_n^{(j)}, \varphi_n^{(j)}\big)$ corresponding to the values of parameters $\mu^{(j)}, k^{(j)}, j = 1, \dots, N$. If $a > 0$ then the lattice (4) admits the breather solutions corresponding to the pairs of complex conjugated points in the discrete spectrum ($\mu^{(1)} = \bar{\mu}^{(2)}, k^{(1)} = \bar{k}^{(2)}$), see figure 1.

2.2. *Nonlinear superposition principle*

The direct recomputing of the variables f is a more convenient way to iterate the Darboux transformation than applying the matrices M and recomputing the wavefunctions. This leads to the nonlinear superposition principle of Darboux transformations in the form of some Yang–Baxter mapping [33]. Let the variables $f_n^{(j)}$ be constructed from the particular solutions of the linear systems at $\mu = \mu^{(j)}$, and let $f_n^{(j,j_1,\dots,j_s)}$ denote the variables obtained from $f_n^{(j)}$ by consecutive applications of Darboux transforms with parameters $\mu^{(j_1)}, \dots, \mu^{(j_s)}$. The permutability of Darboux transformations means that the order of all superscripts except for the first one is not essential and implies the following equality for the matrices of the form (6):

$$M\big(f_n^{(j,k,\sigma)}, \mu^{(j)}\big)M\big(f_n^{(k,\sigma)}, \mu^{(k)}\big) = M\big(f_n^{(k,j,\sigma)}, \mu^{(k)}\big)M\big(f_n^{(j,\sigma)}, \mu^{(j)}\big),$$

where σ stands for a tail sequence of distinct indices. This equation is uniquely solvable with respect to $f_n^{(j,k,\sigma)}, f_n^{(k,j,\sigma)}$ and thus the mapping is defined

$$\begin{gathered}\begin{pmatrix} f_n^{(j,\sigma)} \\ f_n^{(k,\sigma)} \end{pmatrix} \mapsto \begin{pmatrix} f_n^{(j,k,\sigma)} \\ f_n^{(k,j,\sigma)} \end{pmatrix} = \begin{pmatrix} R\big(f_n^{(j,\sigma)}, f_n^{(k,\sigma)}; \mu^{(j)}, \mu^{(k)}\big) \\ R\big(f_n^{(k,\sigma)}, f_n^{(j,\sigma)}; \mu^{(k)}, \mu^{(j)}\big) \end{pmatrix}, \\ R(f, g; \mu, \nu) = \frac{\mu\nu^3(\nu g - \mu f) - a\nu(\mu^2 - \nu^2)fg^2 - a^2(\mu g - \nu f)}{\mu\nu^3(\mu g - \nu f)g + a\nu(\mu^2 - \nu^2) - a^2(\nu g - \mu f)g}.\end{gathered} \tag{10}$$

The direct check proves that the quantity $f_n^{(j,j_1,\dots,j_s)}$ obtained recursively from $f_n^{(j)}, f_n^{(j_1)}, \dots, f_n^{(j_s)}$ is indeed independent of the order of $j_1, \dots, j_n$ (this property is equivalent to the Yang–Baxter equation).

Another formulation of the nonlinear superposition principle brings to a discrete 4-point equation on the square grid for some new variable $z_n^{(j,k)}$ (the subscript corresponds to the shift in the Volterra lattice and is dummy, superscripts enumerate the Darboux transformations). This equation is not too convenient for the purpose of the vector generalizations which we have in mind, however it is of interest by itself and we spend some space to describe it. The form of the equation depends on the sign of a.

In the simplest case $a = 0$ equations (7) imply the relation

$$\frac{\mu}{\tilde{v}_n} - \frac{\mu}{v_{n-1}} = f_{n+1} - f_{n-1}$$

which allows us to introduce the variable z_n according to the equations

$$f_n = \mu(\tilde{z}_n - z_{n-1}), \qquad 1/v_n = z_{n+1} - z_{n-1}.$$

This change turns the relations (7) into a single equation

$$(\tilde{z}_{n+1} - z_n)(z_{n+1} - \tilde{z}_n) = \mu^{-2},$$

which defines Darboux transformation in terms of the variable z. Now, consider another Darboux transformation corresponding to the value $\lambda = \nu$:

$$(\hat{z}_{n+1} - z_n)(z_{n+1} - \hat{z}_n) = \nu^{-2}.$$

The easy calculation proves that the double Darboux transformations coincide: $\hat{\tilde{z}}_n = \tilde{\hat{z}}_n$ and moreover, the common value is given by the superposition formula

$$(\hat{\tilde{z}}_n - z_n)(\hat{z}_n - \tilde{z}_n) = \mu^{-2} - \nu^{-2}.$$

In other words, the Darboux transformations and the superposition formula form the triple which is 3D-consistent, or consistent around a cube [34]. The iterations of Darboux transformation bring to the discrete equation on the square grid (with fixed subscript)

$$\big(z_n^{(j+1,k+1)} - z_n^{(j,k)}\big)\big(z_n^{(j,k+1)} - z_n^{(j+1,k)}\big) = (\mu^{(j)})^{-2} - (\nu^{(k)})^{-2}. \tag{11}$$

This is a very well-known 3D-consistent equation which defines as well the nonlinear superposition principle of the classical Darboux transformation for the Schrödinger operator. This coincidence is not too surprising since it is known for long that Volterra-type lattices are symmetries of the dressing chains which define Bäcklund transformations for KdV-type equations (this relation was discussed, from the different points of view, e.g. in [16–18, 24]).

Analogously, in the case $a = -c^2$ the variable z_n is introduced according to the formulae

$$f_n = \frac{\mu(\tilde{z}_n + z_{n-1})}{c(\tilde{z}_n - z_{n-1})}, \qquad v_n = c\frac{z_{n+1} + z_{n-1}}{z_{n+1} - z_{n-1}}.$$

After this the relations (7) turn into the equation

$$a(\tilde{z}_{n+1} - z_n)(z_{n+1} - \tilde{z}_n) = -\mu^2(\tilde{z}_{n+1} + z_n)(z_{n+1} + \tilde{z}_n),$$

and (11) is replaced by the equation

$$r(\mu^{(j)})\big(z_n^{(j,k)} z_n^{(j,k+1)} + z_n^{(j+1,k)} z_n^{(j+1,k+1)}\big) = r(\nu^{(k)})\big(z_n^{(j,k)} z_n^{(j+1,k)} + z_n^{(j,k+1)} z_n^{(j+1,k+1)}\big), \tag{12}$$

where $r(\lambda) = (\lambda^2 - a)/(\lambda^2 + a)$, which is equivalent to the nonlinear superposition principle for the sinh-Gordon equation.

Finally, if $a = c^2$ then the change

$$f_n = \frac{\mu(1 + z_{n-1}\tilde{z}_n)}{c(z_{n-1} - \tilde{z}_n)}, \qquad v_n = c\frac{1 + z_{n+1}z_{n-1}}{z_{n+1} - z_{n-1}}$$

is used which brings equations (7) to the form

$$a(\tilde{z}_{n+1} - z_n)(z_{n+1} - \tilde{z}_n) = \mu^2(1 + \tilde{z}_{n+1}z_n)(1 + z_{n+1}\tilde{z}_n)$$

and leads to the equation

$$\begin{aligned}(r(\mu^{(j)}) + r(\nu^{(j)}))&\big(z_n^{(j,k+1)} - z_n^{(j+1,k)}\big)\big(z_n^{(j,k)} - z_n^{(j+1,k+1)}\big)\\ &= (r(\mu^{(j)}) - r(\nu^{(j)}))\big(1 + z_n^{(j,k+1)} z_n^{(j+1,k)}\big)\big(1 + z_n^{(j,k)} z_n^{(j+1,k+1)}\big)\end{aligned} \tag{13}$$

equivalent to the nonlinear superposition principle for the sine-Gordon equation. Equation (12) turns into (13) under the complex change $z \to (\mathrm{i} - z)/(\mathrm{i} + z)$, so that these equations are two different real forms of one and the same equation.

Concluding this section, we note that an analogous construction scheme exists also for solutions of the Volterra lattice

$$u_{n,x} = u_n(u_{n+1} - u_{n-1}).$$

The corresponding formulae are even much simpler, for example the equations

$$u_n = (v_n - \mu)(v_{n+1} + \mu), \qquad \tilde{u}_n = (v_{n+1} - \mu)(v_n + \mu)$$

replace (7) while the role of the lattice (9) is played by the lattice (4) at $a = -\mu^2$. Therefore, the lattice (9) is actually the second modification of Volterra lattice. This sequence is analogous to the sequence of equations KdV → mKdV → exp-CD (exponential Calogero–Degasperis equation) which can be obtained by continuous limit from the lattices under consideration. Unfortunately, although the Volterra lattice admits some multi-component generalizations [7, 35], the vector ones are absent, this is why we have started from the more complicated object.

3. First vector generalization

Sometimes the zero curvature representation for a vector generalization can be obtained just by passing to the block matrices. Unfortunately, this is not the case for the matrices (5) and (6). It turns out, however, that such block generalization is easy if one considers the linear equations for the three-dimensional vector with the components consisting of the products of the components of Ψ. Additionally, it is convenient to apply a gauge transformation in order to make the determinants of the matrices L, M constant and the matrix A traceless. In this way we come to the following matrices which define, as can be easily verified, the zero curvature representation for the lattice (4) at $a = 0$ and its Bäcklund transformation:

$$L_n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & \frac{\lambda}{v_n} \\ 1 & -\frac{2\lambda}{v_n} & \frac{\lambda^2}{v_n^2} \end{pmatrix}, \qquad M_n = \begin{pmatrix} \frac{\lambda^2}{\mu^2 f_n^2} & -\frac{2\lambda}{\mu f_n} & 1 \\ \frac{\lambda}{\mu f_n} & -1-\frac{\lambda^2}{\mu^2} & \frac{\lambda f_n}{\mu} \\ 1 & -\frac{2\lambda f_n}{\mu} & \frac{\lambda^2 f_n^2}{\mu^2} \end{pmatrix},$$

$$A_n = \lambda \begin{pmatrix} -\lambda & 2v_{n-1} & 0 \\ -v_n & 0 & v_{n-1} \\ 0 & -2v_n & \lambda \end{pmatrix}.$$

It is not difficult to find the matrices for the general case $a \neq 0$, but they are more cumbersome. Fortunately, we will not need them, since one of the vector lattices exists only in the case $a = 0$ anyway (see section 6), and for the second one this assumption does not lead to the loss of generality (see section 4).

The block matrices for the vector lattices are derived from here under the 'proper' interpretation of v_n as a vector-valued quantity. To make notation more clear we write vectors in the upper case. We assume that the vector space is equipped with a symmetric scalar product $\langle U, V\rangle = \langle V, U\rangle$. The identity operator is denoted by I and the linear form $V^\top$, inverse vector V^{-1} and operator $UV^\top$ are defined as follows:

$$V^\top(U) = \langle V, U\rangle, \qquad V^{-1} = \frac{V}{\langle V, V\rangle} \qquad UV^\top(W) = U\langle V, W\rangle.$$

In the case of finite-dimensional Euclidean space one can think of V as the column vector and of $V^\top$ as the row vector.

The first vector analog of the lattice (4) exists only at $a = 0$. It is of the form [6]

$$V_{n,x} = 2\langle V_n, V_{n+1} - V_{n-1}\rangle V_n - \langle V_n, V_n\rangle(V_{n+1} - V_{n-1}). \tag{14}$$

This lattice appears as the compatibility condition for the linear systems

$$T\begin{pmatrix} \psi_{n-1} \\ \Psi_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I & \lambda V_n^{-1} \\ 1 & -2\lambda\left(V_n^{-1}\right)^\top & \lambda^2/\langle V_n, V_n\rangle \end{pmatrix}\begin{pmatrix} \psi_{n-1} \\ \Psi_n \\ \psi_n \end{pmatrix}, \tag{15}$$

$$D_x\begin{pmatrix} \psi_{n-1} \\ \Psi_n \\ \psi_n \end{pmatrix} = \lambda\begin{pmatrix} -\lambda & 2V_{n-1}^\top & 0 \\ -V_n & 0 & V_{n-1} \\ 0 & -2V_n^\top & \lambda \end{pmatrix}\begin{pmatrix} \psi_{n-1} \\ \Psi_n \\ \psi_n \end{pmatrix}. \tag{16}$$

Note that the systems (15) and (16) possess the first integral in common

$$J = \langle \Psi_n, \Psi_n\rangle - \psi_n\psi_{n-1}, \qquad (T-1)(J) = D_x(J) = 0. \tag{17}$$

The Darboux transformation is defined by a particular solution at the zero level of this first integral.

Statement 1. *Let* $F_n = \Phi_n/\phi_n$ *where* $\psi = \phi, \Psi = \Phi$ *is a particular solution of the linear systems (15) and (16) at* $\lambda = \mu$ *and at* $J = 0$*. Then the transformation*

$$\begin{pmatrix} \tilde\psi_{n-1} \\ \tilde\Psi_n \\ \tilde\psi_n \end{pmatrix} = \begin{pmatrix} \frac{\lambda^2}{\mu^2\langle F_n, F_n\rangle} & -\frac{2\lambda}{\mu}\left(F_n^{-1}\right)^\top & 1 \\ \frac{\lambda}{\mu}F_n^{-1} & \left(\frac{\lambda^2}{\mu^2}-1\right)I - \frac{2\lambda^2}{\mu^2}F_n^{-1}F_n^\top & \frac{\lambda}{\mu}F_n \\ 1 & -\frac{2\lambda}{\mu}F_n^\top & \frac{\lambda^2}{\mu^2}\langle F_n, F_n\rangle \end{pmatrix} \begin{pmatrix} \psi_{n-1} \\ \Psi_n \\ \psi_n \end{pmatrix} \qquad (18)$$

maps the general solution of these systems into the solution of the systems of the same form, with the original and transformed potentials related by the equations

$$\mu V_n^{-1} = F_n + F_{n+1}^{-1}, \qquad \mu\tilde V_n^{-1} = F_{n+1} + F_n^{-1}. \qquad (19)$$

The expanded form of relations (19) is

$$\begin{aligned} V_n &= \mu\frac{F_{n+1} + \langle F_{n+1}, F_{n+1}\rangle F_n}{1 + 2\langle F_n, F_{n+1}\rangle + \langle F_n, F_n\rangle\langle F_{n+1}, F_{n+1}\rangle}, \\ \tilde V_n &= \mu\frac{F_n + \langle F_n, F_n\rangle F_{n+1}}{1 + 2\langle F_n, F_{n+1}\rangle + \langle F_n, F_n\rangle\langle F_{n+1}, F_{n+1}\rangle}. \end{aligned} \qquad (20)$$

Each of these transformations is the substitution into the lattice (14) from the lattice

$$F_{n,x} = \mu^2\left(F_{n-1} + F_n^{-1}\right)^{-1} - \mu^2\left(F_{n+1} + F_n^{-1}\right)^{-1}.$$

It is easy to see that these formulae turn into (7) and (9) in the scalar case at $a = 0$.

The derivation of the nonlinear superposition principle is not more difficult than in the scalar case. The following Yang–Baxter mapping (cf equation 10 at $a = 0$) can be obtained by multiplying the matrices M of the form (18):

$$\begin{aligned} &F_n^{(j,k,\sigma)} = R\left(F_n^{(j,\sigma)}, F_n^{(k,\sigma)}; \mu^{(j)}, \mu^{(k)}\right), \qquad F_n^{(k,j,\sigma)} = R\left(F_n^{(k,\sigma)}, F_n^{(j,\sigma)}; \mu^{(k)}, \mu^{(j)}\right), \\ &R(F, G; \mu, \nu) = \frac{(\mu^2 - \nu^2)\langle G, G\rangle F + \mu(\nu\langle F, F\rangle - 2\mu\langle F, G\rangle + \nu\langle G, G\rangle)G}{\langle G, G\rangle\langle \nu F - \mu G, \nu F - \mu G\rangle}. \end{aligned} \qquad (21)$$

It is possible to obtain the analog of equation (11), too. Let us introduce the new vector variable Z_n according to the formulae

$$F_n = \mu(\tilde Z_n - Z_{n-1}), \qquad V_n^{-1} = Z_{n+1} - Z_{n-1}.$$

Equations (19) become equivalent to the single equation

$$\tilde Z_{n+1} - Z_n = \mu^{-2}(Z_{n+1} - \tilde Z_n)^{-1}$$

under this change. Next, consider the Darboux transformation corresponding to the spectral value $\lambda = \nu$:

$$\hat Z_{n+1} - Z_n = \nu^{-2}(Z_{n+1} - \hat Z_n)^{-1}.$$

The direct calculation shows that the repeated Darboux transformations coincide: $\hat{\tilde Z}_n = \tilde{\hat Z}_n$ and moreover, the result is given by the equation

$$\hat{\tilde Z}_n - Z_n = (\mu^{-2} - \nu^{-2})(\hat Z_n - \tilde Z_n)^{-1}.$$

Iterations of the Darboux transformation are governed by the 3D-consistent discrete equation on the square grid (the subscript n is dummy):

$$Z_n^{(j+1,k+1)} - Z_n^{(j,k)} = ((\mu^{(j)})^{-2} - (\nu^{(k)})^{-2})\left(Z_n^{(j,k+1)} - Z_n^{(j+1,k)}\right)^{-1}.$$

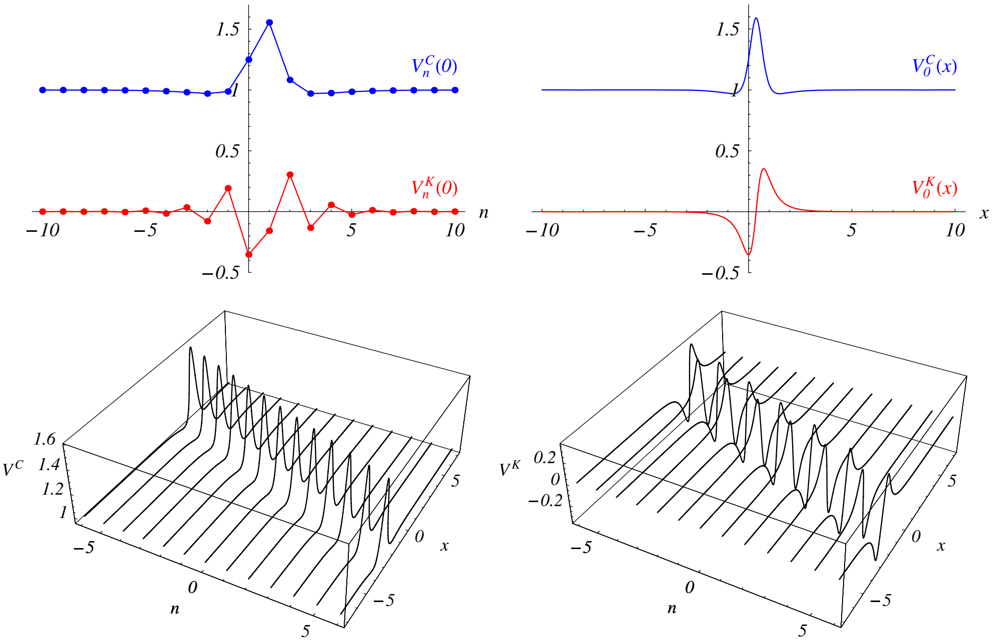

Figure 2. A soliton of the lattice (14); $C = (1, 0)$, $K = (0, 1)$, $\gamma = 2$, $c_1 = 1$, $c_3 = -1$.

This equation with important applications in the discrete geometry was introduced in [19] (a special reduction was considered in [22]), see also [20, 21].

Let us make use of Darboux transformation for the construction of the soliton solution. The solution of the linear equations (15) and (16) with constant coefficients $V_n = C = \text{const}$, $\langle C, C\rangle = 1$, at $\lambda = \mu$ reads

$$\begin{aligned}
&\phi_n = c_1\gamma^n\,\mathrm{e}^{(\gamma-\gamma^{-1})x} + c_2\gamma^{-n}\,\mathrm{e}^{(\gamma^{-1}-\gamma)x} + 2c_3,\\
&\Phi_n = (-1)^n K + \mu\left(\frac{c_1\gamma^n}{1+\gamma}\,\mathrm{e}^{(\gamma-\gamma^{-1})x} + \frac{c_2\gamma^{-n}}{1+\gamma^{-1}}\,\mathrm{e}^{(\gamma^{-1}-\gamma)x} + c_3\right)C, \qquad (22)\\
&\mu^2 = \gamma + 2 + \gamma^{-1}, \qquad \langle C, K\rangle = 0, \qquad \gamma\langle K, K\rangle = (1-\gamma)^2\big(c_1c_2 - c_3^2\big)
\end{aligned}$$

(the latter relation is equivalent to the constraint $J = 0$; and we do not consider the cases of multiple eigenvalues $\mu = 0$, $\mu = \pm 2$). Equations (19) bring, after elementary transformations, to the one-soliton solution of the lattice (figure 2)

$$\tilde{V}_n = \frac{\phi_{n+1}\Phi_n + \phi_{n-1}\Phi_{n+1}}{\mu\phi_n^2}.$$

Clearly, this solution always lies in the plane of the vectors C, K, that is it is actually a 2-component, independently of the dimension of the vector space under consideration. The C-component is a soliton on the unit background. Its shape is slightly different for positive and negative values of c_3. The K-component has localized oscillations on the zero background which originate from the powers of -1 in the solution (22). In contrast to the scalar case at $a = 0$, the additional dimension also makes possible the breathers corresponding to the complex conjugated discrete spectrum (figure 3). The N-soliton solution is constructed by applying the map (21) to the vectors $F_n^{(j)} = \Phi_n^{(j)}\big/\phi_n^{(j)}$ where $\big(\phi_n^{(j)}, \Phi_n^{(j)}\big)$ are solutions of the

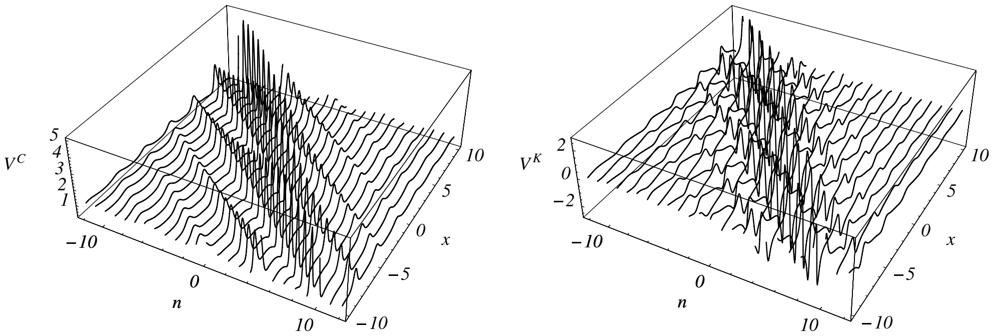

Figure 3. A breather of the lattice (14); $C = (1, 0)$, $K = (0, 1)$, $\gamma^{(1)} = \bar\gamma^{(2)} = 0.5 + i$, $c_1^{(1)} = c_1^{(2)} = 0.5$, $c_3^{(1)} = c_3^{(2)} = 1$.

form (22) corresponding to the values of parameters $\gamma^{(j)}, c_1^{(j)}, c_3^{(j)}$ and $K^{(j)}$, $j = 1, \dots, N$. This solution evolves in the space spanned over the vectors $C, K^{(1)}, \dots, K^{(N)}$.

4. Second vector generalization

The lattice

$$U_{n,x} = (\langle U_n, U_n\rangle + a)(U_{n+1} - U_{n-1}) \tag{23}$$

is, in contrast to (14), integrable at an arbitrary value of parameter a though it turns out to be not so important as in the scalar case. Indeed, it can easily be eliminated or, more precisely, 'confined inside the lattice' at the expense of increasing by 1 the dimension of the vector space under consideration. This is done by means of the orthogonal complement: let U_n be a solution of the lattice (23) then the vector

$$V_n = U_n + E, \qquad E = \text{const}, \qquad \langle U_n, E\rangle = 0, \qquad \langle E, E\rangle = a \tag{24}$$

(if $a < 0$ then a pseudoeuclidean scalar product is used) satisfies the lattice

$$V_{n,x} = \langle V_n, V_n\rangle (V_{n+1} - V_{n-1}). \tag{25}$$

This transformation does not lead to any problem when constructing solutions since the reduction (24) is consistent with higher symmetries and Bäcklund transformation. On the other hand, it essentially simplifies all formulae (cf e.g., equations (29) and (33)). The matrices of the zero curvature representation become simpler as well.

The lattice (25) is the compatibility condition of the linear systems

$$T\begin{pmatrix}\psi_{n-1}\\ \Psi_n\\ \psi_n\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & I - 2V_n^{-1}V_n^\top & \lambda V_n^{-1}\\ 1 & -2\lambda\left(V_n^{-1}\right)^\top & \lambda^2/\langle V_n, V_n\rangle\end{pmatrix}\begin{pmatrix}\psi_{n-1}\\ \Psi_n\\ \psi_n\end{pmatrix}, \tag{26}$$

$$D_x\begin{pmatrix}\psi_{n-1}\\ \Psi_n\\ \psi_n\end{pmatrix} = \begin{pmatrix}-\lambda^2 & 2\lambda V_{n-1}^\top & 0\\ -\lambda V_n & 2V_nV_{n-1}^\top - 2V_{n-1}V_n^\top & \lambda V_{n-1}\\ 0 & -2\lambda V_n^\top & \lambda^2\end{pmatrix}\begin{pmatrix}\psi_{n-1}\\ \Psi_n\\ \psi_n\end{pmatrix}. \tag{27}$$

These systems possess the first integral (17) in common, as in the previous case.

Statement 2. *Let $F_n = \Phi_n/\phi_n$ where $\psi = \phi, \Psi = \Phi$ is a particular solution of the linear systems (26) and (27) at $\lambda = \mu$ and such that $J = \langle\Phi_n, \Phi_n\rangle - \phi_n\phi_{n-1} = 0$. Then the transform*

$$\begin{pmatrix} \tilde\psi_{n-1} \\ \tilde\Psi_n \\ \tilde\psi_n \end{pmatrix} = \begin{pmatrix} \frac{\lambda^2}{\mu^2\langle F_n, F_n\rangle} & -\frac{2\lambda}{\mu}\left(F_n^{-1}\right)^\top & 1 \\ \frac{\lambda}{\mu}F_n^{-1} & \left(1-\frac{\lambda^2}{\mu^2}\right)I - 2F_n^{-1}F_n^\top & \frac{\lambda}{\mu}F_n \\ 1 & -\frac{2\lambda}{\mu}F_n^\top & \frac{\lambda^2}{\mu^2}\langle F_n, F_n\rangle \end{pmatrix} \begin{pmatrix} \psi_{n-1} \\ \Psi_n \\ \psi_n \end{pmatrix} \tag{28}$$

maps the general solution of these systems into the solution of the systems of the same kind, with the original and transformed potentials related by the equations

$$V_n = \mu\frac{F_{n+1} - \langle F_{n+1}, F_{n+1}\rangle F_n}{1 - \langle F_n, F_n\rangle\langle F_{n+1}, F_{n+1}\rangle}, \qquad \tilde V_n = \mu\frac{F_n - \langle F_n, F_n\rangle F_{n+1}}{1 - \langle F_n, F_n\rangle\langle F_{n+1}, F_{n+1}\rangle}. \tag{29}$$

In comparison with the previous section, equations (29) are slightly shorter than (20), but the analog of the lattice (9) is more cumbersome:

$$F_{n,x} = \frac{\mu^2\langle F_n, F_n\rangle\left(\left(F_n - F_{n-1}^{-1}\right)^{-1} - \left(F_n - F_{n+1}^{-1}\right)^{-1}\right)}{\left\langle\left(F_n - F_{n-1}^{-1}\right)^{-1}, F_n + F_{n-1}^{-1}\right\rangle\left\langle\left(F_n - F_{n+1}^{-1}\right)^{-1}, F_n + F_{n+1}^{-1}\right\rangle}.$$

The superposition of Darboux transformations is defined by the Yang–Baxter map

$$F_n^{(j,k,\sigma)} = R\left(F_n^{(j,\sigma)}, F_n^{(k,\sigma)}; \mu^{(j)}, \mu^{(k)}\right), \qquad F_n^{(k,j,\sigma)} = R\left(F_n^{(k,\sigma)}, F_n^{(j,\sigma)}; \mu^{(k)}, \mu^{(j)}\right),$$
$$R(F, G; \mu, \nu) = \frac{(\nu^2 - \mu^2)\langle G, G\rangle F + \nu(\mu\langle F, F\rangle - 2\nu\langle F, G\rangle + \mu\langle G, G\rangle)G}{\langle G, G\rangle\langle\nu F - \mu G, \nu F - \mu G\rangle}. \tag{30}$$

Note that the formulae (21) and (30) coincide up to the permutation of μ and ν in the numerator. Despite such similarity, an analog of equation (11) is probably lacking in this case.

Statement 3. *The Darboux transformation is consistent with the reduction (24).*

Proof. Let us apply the change $V_n = U_n + E$, $F_n = H_n + h_nE$, $\langle H_n, E\rangle = 0$ to the equations (29), with the scalar factor h_n unknown for the moment:

$$U_n + E = \mu\frac{H_{n+1} + h_{n+1}E - \left(\langle H_{n+1}, H_{n+1}\rangle + ah_{n+1}^2\right)(H_n + h_nE)}{1 - \left(\langle H_n, H_n\rangle + ah_n^2\right)\left(\langle H_{n+1}, H_{n+1}\rangle + ah_{n+1}^2\right)},$$
$$\tilde U_n + E = \mu\frac{H_n + h_nE - \left(\langle H_n, H_n\rangle + ah_n^2\right)(H_{n+1} + h_{n+1}E)}{1 - \left(\langle H_n, H_n\rangle + ah_n^2\right)\left(\langle H_{n+1}, H_{n+1}\rangle + ah_{n+1}^2\right)}. \tag{31}$$

Collecting the coefficients of E yields the coupled algebraic equations for h_n and h_{n+1}. It is not obvious beforehand that their solution is compatible with the shift in n. If this would not be the case then the change $F_n = H_n + h_nE$ would be incorrect. However, the direct computation proves that h_n is defined by one and the same formula for all n as a solution of the quadratic equation

$$ah_n^2 - \mu h_n + \langle H_n, H_n\rangle + 1 = 0, \tag{32}$$

therefore the E-component is detached in the transformation (29). □

Statement 3 makes redundant the separate study of the case $a \neq 0$. Nevertheless, all formulae can be, in principle, rewritten for this case as well, moreover, their rational structure can be preserved by the use of the stereographic projection for the quadric (32):

$$H_n = \frac{(\nu^2 - a)F_n}{\nu^2 + a\langle F_n, F_n\rangle}, \qquad h_n = \frac{\nu(1 + \langle F_n, F_n\rangle)}{\nu^2 + a\langle F_n, F_n\rangle}, \qquad \mu = \nu + \frac{a}{\nu}$$

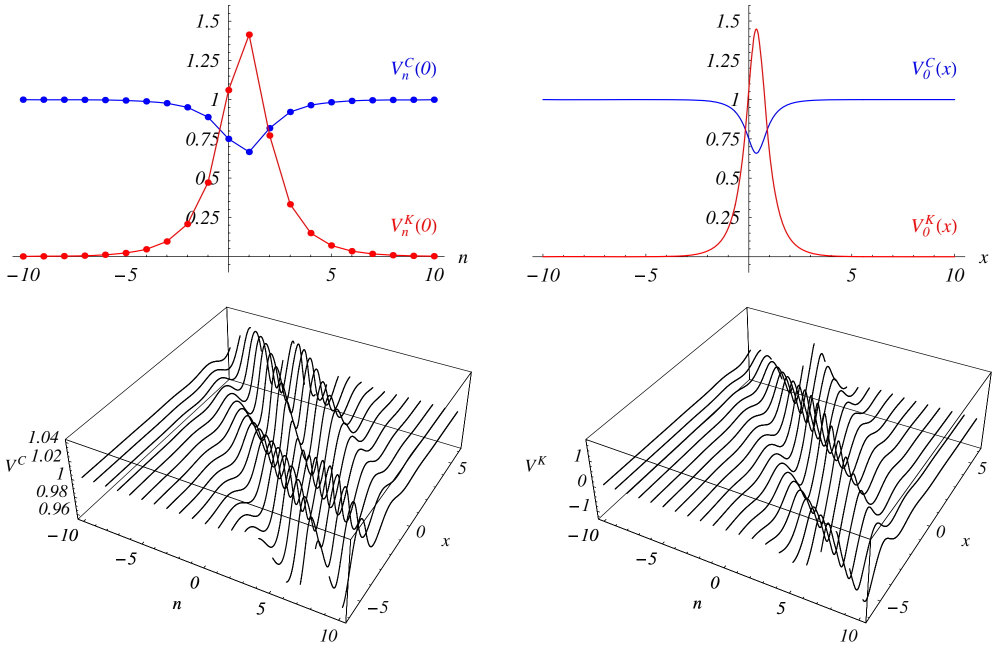

Figure 4. Solutions of the lattice (25) at $C = (1, 0)$, $K = (0, 1)$. Soliton: $\gamma = 2$, $c_1 = 1$, $c_3 = -1$; breather: $\gamma^{(1)} = \bar{\gamma}^{(2)} = 1 + 1.5\mathrm{i}$, $c_1^{(1)} = c_1^{(2)} = 3$, $c_3^{(1)} = c_3^{(2)} = -1$.

(more rigorously, some other letter should be used here instead of F, but we hope it will not lead to misunderstanding). For instance, the substitution into (31) brings, under this parametrization, to the Bäcklund transformation for the lattice with parameter (23):

$$\begin{aligned} U_n &= \frac{(\nu^2 + a\langle F_n, F_n\rangle)F_{n+1} - (a + \nu^2\langle F_{n+1}, F_{n+1}\rangle)F_n}{\nu(1 - \langle F_n, F_n\rangle\langle F_{n+1}, F_{n+1}\rangle)}, \\ \tilde{U}_n &= \frac{(\nu^2 + a\langle F_{n+1}, F_{n+1}\rangle)F_n - (a + \nu^2\langle F_n, F_n\rangle)F_{n+1}}{\nu(1 - \langle F_n, F_n\rangle\langle F_{n+1}, F_{n+1}\rangle)}. \end{aligned} \tag{33}$$

The Yang–Baxter map (30) can be rewritten in a more general form in a similar way. The transformation (33) turns into (29) at $a = 0$, while in the scalar case we come back to the transformation (7), under identifying U, F, ν with v, f, μ respectively.

It is not difficult to compute, by the use of (29), the one-soliton solution

$$\tilde{V}_n = \mu \frac{\phi_{n+1}\Phi_n - \phi_{n-1}\Phi_{n+1}}{\phi_n(\phi_{n+1} - \phi_{n-1})}$$

where the solution ϕ_n, Φ_n of equations (26) and (27) with constant coefficients is given by almost the same formulae (22) as before, with the only difference being that the factor $(-1)^n$ in front of K disappears. The construction of multisoliton and breather solutions is quite analogous to the previous case.

The multisoliton solutions of the lattice (23) with $a = 1$ and $U_n \to 0$, $n \to \pm\infty$ were constructed in the paper [15] by the use of ISTM for a spectral problem associated with matrices of $2^d \times 2^d$ size where d is the dimension of the vector U_n. These solutions give rise, through the transformation (24), to some class of solutions of the lattice (25) with boundary conditions $\langle V_n, V_n\rangle \to 1$, $n \to \pm\infty$. On the other hand, statement 3 says that the form (24) can be preserved by the Darboux transformation (29) under the special reduction expressed by equation (32). Therefore, Darboux transformations can be used for the construction of the solutions of the lattice (23) with vanishing boundary conditions. The question naturally arises regarding the equivalence of these solution classes. The difference between the spectral problems makes the comparison of the results difficult, but the following simple argument shows at least that there exist solutions which are not equivalent to those obtained in [15].

The first two plots of the figure 4 suggest that the *C*- and *K*-components of the one-soliton solution $V_n = V_n^C C + V_n^K K$ are related by the constraint $V_n^C - 1 = \alpha V_n^K$ with some constant α, and this is indeed the case, as one can check by direct computation. This allows to set $V_n^C = 1$ by the use of some orthogonal transformation, so that V_n is of the form (24) with $a = 1$ and the one-dimensional vector U_n. Effectively, this means that any one-soliton solution of the lattice (25) is equivalent to one-soliton solution of the *scalar* lattice (23). However, the generic two-soliton solution of the lattice (25) cannot be brought to the form (24) by any orthogonal transform. This can easily be verified numerically: it is sufficient to check, at random values of parameters, that the coefficients of the expansion of two-soliton solution over the base $C, K^{(1)}, K^{(2)}$ are affine-linearly independent.

5. Higher symmetries and associated systems

Both vector lattices (14) and (25) belong to an infinite hierarchy of commuting flows. We restrict ourselves by consideration of the simplest higher symmetries which are of the second order with respect to the shift in n. In the scalar case one has, setting $a = 0$ for simplicity, the pair of consistent lattices

$$v_{n,x} = v_n^2(v_{n+1} - v_{n-1}), \qquad v_{n,t} = v_n^2\big(v_{n+1}^2(v_{n+2} + v_n) - v_{n-1}^2(v_n + v_{n-2})\big). \tag{34}$$

Obviously, the first of these equations can be solved with respect to v_{n+1} or v_{n-1}, and this allows us to express recursively all v_j through the pair of variables $u = v_{n+1}$, $v = v_n$. After this, the symmetry takes the form of Kaup–Newell evolution system

$$u_t = u_{xx} + (2u^2 v)_x, \qquad v_t = -v_{xx} + (2uv^2)_x$$

and the shift in n defines an explicit auto-substitution for this system (the simplest type of Bäcklund transforms).

The lattice (14) and its symmetry can be compactly written in the form preserving the structure (34)

$$V_{n,x} = P_{V_n}(V_{n+1} - V_{n-1}), \qquad V_{n,t} = P_{V_n}(P_{V_{n+1}}(V_{n+2} + V_n) - P_{V_{n-1}}(V_n + V_{n-2})) \tag{35}$$

by the use of the operator $P_V(U) = 2\langle V, U\rangle V - \langle V, V\rangle U$. It is easy to check that the identity $(P_V)^{-1} = P_{V^{-1}}$ is valid for this operator. Making use of it one can solve, like before, the first equation with respect to V_{n+1} or V_{n-1} and express all V_j through the pair of variables $U = V_{n+1}$, $V = V_n$. This brings the symmetry to the form of the vector generalization of the Kaup–Newell system

$$U_t = U_{xx} + (4\langle U, V\rangle U - 2\langle U, U\rangle V)_x, \qquad V_t = -V_{xx} + (4\langle U, V\rangle V - 2\langle V, V\rangle U)_x. \tag{36}$$

Analogously, the commuting flows for the second vector lattice are

$$\begin{aligned}
V_{n,x} &= \langle V_n, V_n\rangle(V_{n+1} - V_{n-1}),\\
V_{n,t} &= \langle V_n, V_n\rangle\big(\langle V_{n+1}, V_{n+1}\rangle(V_{n+2} - V_n) + \langle V_{n-1}, V_{n-1}\rangle(V_n - V_{n-2})\\
&\quad + 2(\langle V_{n+1}, V_n\rangle + \langle V_n, V_{n-1}\rangle)(V_{n+1} - V_{n-1})\big)
\end{aligned} \tag{37}$$

and the associated evolution system reads

$$U_t = U_{xx} + 4\langle U, V\rangle U_x + 2\langle U, U\rangle V_x, \qquad V_t = -V_{xx} + 4\langle U, V\rangle V_x + 2\langle V, V\rangle U_x \tag{38}$$

which is another vector analog of the Kaup–Newell system.

Another interesting type of associated systems is obtained for the scalar quantities

$$p_n = \langle V_n, V_n\rangle, \qquad q_n = 2\langle V_n, V_{n-1}\rangle$$

which satisfy, in virtue of any of the pair (35) or (37) one and the same two-dimensional modified Volterra lattice

$$p_{n,t} + 2p_n^2(p_{n+1} - p_{n-1}) = p_n\big((q_{n+1} + q_n)_x + q_{n+1}^2 - q_n^2\big), \qquad p_{n,x} = p_n(q_{n+1} - q_n). \tag{39}$$

It can be written in the form

$$p_{n,t} + 2p_n^2(p_{n+1} - p_{n-1}) = (r_n p_n)_x, \qquad (p_{n+1}p_n)_x = p_{n+1}p_n(r_{n+1} - r_n),$$

as well, where $r_n = q_{n+1} + q_n = 2\langle V_n, V_{n+1} + V_{n-1}\rangle$. These lattices are closely related to Mikhailov lattices introduced in [25].

The lattices (35) and (37) can be effectively used for the construction of particular solutions of the systems (36) and (38) and the lattice (39). Along with the construction method of the soliton-type solutions described above, one can use to this end the periodic closure $V_{n+N} = CV_n$ with orthogonal operator C which leads to the finite-dimensional dynamical systems.

6. Further vector analogs

Recall that the classification problem of scalar integrable lattices of Volterra type was solved by Yamilov [27] within the symmetry approach. Recently, one of the authors has obtained an analogous classification of the vector Volterra lattices on the sphere, that is under the constraint $\langle V_n, V_n\rangle = 1$ [32]. This constraint essentially simplifies the problem which is very complicated and remains open for the case of free space. Other simplifying assumptions can be used of course, for example the polynomiality of the lattice. It should be noted that in the continuous case many polynomial equations are known; we mention only the papers [1–3] which contain the examples and some classification results for the vector systems of derivative nonlinear Schrödinger type, equations (36) and (38) being just two instances of such systems. In the discrete case, however, the polynomiality is not a too natural assumption, as one can see already from the Yamilov list of scalar lattices. In the vector setting we have not succeeded in finding other polynomial Volterra-type lattices possessing higher symmetries aside from (14) and (23).

Our search for integrable lattices was based on the straightforward method of undetermined coefficients. In the simplest case the lattice and its symmetry are of the form

$$V_{n,x} = a^{(1)}V_{n+1} + a^{(0)}V_n + a^{(-1)}V_{n-1}, \qquad V_{n,t} = b^{(2)}V_{n+2} + \cdots + b^{(-2)}V_{n-2}$$

where the scalar coefficients $a^{(i)}$ are linear with respect to the scalar products of V_{n+1}, V_n, V_{n-1} and $b^{(i)}$ are quadratic with respect to the scalar products of $V_{n+2}, \ldots, V_{n-2}$. It is easy to find that the homogeneous lattice contains 18 parameters and its symmetry contains 600 ones. Calculating the cross derivatives yields a system of bilinear equations for the coefficients. Although this system is very bulky, its solving is, in principle, not difficult since the equations are very over determined and sparse (in particular, a large number of equations are monomial). The answer is the consistent pairs of the lattices (35) and (37) (there are also few solutions with $a^{(1)} = a^{(-1)} = 0$, but all such lattices can be reduced to the scalar ones and therefore they are not of interest to us). Adding the lower degree terms to the lattice and its symmetry allows us to introduce the parameter a into the pair (37) and to prove that no such generalization exists for the pair (35).

It is clear that the scope of this method in this problem is very much restricted. If one takes the coefficients $a^{(i)}$ quadratic with respect to the scalar products and $b^{(i)}$ of the fourth degree then the number of unknown parameters in the lattice and its symmetry becomes 63 and 15 300 respectively, and even the calculation of the commutator becomes a difficult task. This case is still manageable, but with the empty answer.

We also have partially analyzed the case when the lattice is of the second order with respect to the shift in n and its symmetry is of the fourth order, that is

$$V_{n,x} = a^{(2)} V_{n+2} + \cdots + a^{(-2)} V_{n-2}, \qquad V_{n,t} = b^{(4)} V_{n+4} + \cdots + b^{(-4)} V_{n-4}.$$

One may hope that some vector analogs of the Narita–Bogoyavlensky lattice [35–37] appear here, more precisely, analogs of some its modification with odd degree of nonlinearity, for example

$$v_{n,x} = v_n(v_{n+2}v_{n+1} - v_{n-1}v_{n-2}) \qquad \text{or} \qquad v_{n,x} = v_{n+1}v_n^3 v_{n-1}(v_{n+2}v_{n+1} - v_{n-1}v_{n-2}).$$

Note that classification of such lattices is not known even in the scalar case. Unfortunately, the analogs of the Narita–Bogoyavlensky lattice have not been discovered; however we have found two more lattices relative to the Volterra lattice:

$$V_{n,x} = \langle V_n, V_n\rangle(\langle V_{n+1}, V_{n+1}\rangle(V_{n+2} + V_n) - \langle V_{n-1}, V_{n-1}\rangle(V_n + V_{n-2})), \tag{40}$$

$$V_{n,x} = \langle V_{n+1}, V_n\rangle\langle V_n, V_{n-1}\rangle(V_{n+2} - V_{n-2}). \tag{41}$$

Both lattices possess fourth order symmetries which we do not bring because of their length. The study of these examples falls beyond the scope of our paper. We only notice that the lattice (40) generalizes the second flow of the modified Volterra lattice (34), so that this flow admits at least three vector analogs. The question of the number of vector analogs for the higher flows of the hierarchy remains open. The lattice (41) in the scalar case is a modification of the Volterra lattice on the 'stretched' grid:

$$v_{n,x} = v_{n+1}v_n^2 v_{n-1}(v_{n+2} - v_{n-2}) \xrightarrow{u_n = v_{n+2}v_{n+1}v_n v_{n-1}} u_{n,x} = u_n(u_{n+2} - u_{n-2}),$$

but in the vector case this substitution makes no sense and the lattice (41) seems to be an independent object. The zero curvature representations and Bäcklund transformations for the lattices (40) and (41) are not known for now.

Summing up, we may say that the classification of the polynomial lattices of Volterra and Narita–Bogoyavlensky types is a very difficult open problem, probably with very scarce answers. The alternative approaches to the method of undetermined coefficients are the analysis of the necessary integrability conditions in the form of canonical conservation laws [27–29] and the perturbative approach [38], however the contemporary state of the theory does not allow us to effectively apply them, even in the scalar case.

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